Exam. Code : 211003
Subject Code : 3848

M.Sc. Mathematics 3rd Semester
FUNCTIONAL ANALYSIS—I
Paper—MATH-571

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt any **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

1. If M is a closed linear subspace of a Banach space N prove that the quotient space $\frac{N}{M}$ is complete (specify the norm on $\frac{N}{M}$).

   Show by an example that $\frac{N}{M}$ may not be a normed linear space if M is not closed.

2. Prove or disprove that the space C[0, 1] with

   $$\|f\| = \int_0^1 |f(x)|\, dx \text{ is complete.}$$

3. If p and q are nonnegative extended real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ and if $f \in L^p$, $g \in L^q$ then show, that $f_g \in L_1$ and $\int |f_g| \le \|f\|_p \|g\|_q$.

4. Define L^p space and show that

   L^p $(1 \leq p < \infty)$ is complete.

UNIT—II

5. Let T be a linear transformation of normed linear space N into normed linear space N′. Show that T is continuous iff the image T(S) is bounded in N′ where $S = \{x : \| x \| \leq 1\}$ is the closed unit sphere in N.

6. Prove that all norms on $\mathbb{R}^n$ are equivalent.

7. Prove that every linear transformation whose domain is finite dimensional normed linear space, is continuous. Does the result hold true if the space is infinite dimensional. Justify.

8. If a normed space X has the property that all closed bounded sets in X are compact, show that X is finite dimensional.

UNIT—III

9. If x and y are distinct elements of normed linear space N, then there exists f in N* such that

   $f(x) \neq f(y)$.

10. Show that for a normed linear space N, if N* is separable then N is separable. Does the converse hold true ?

11. Show that there exists an isometric isomorphism of a normed linear space N into its second conjugate space N**.

12. Prove that every finite dimensional normed linear space is reflexive.

Justify the statement that a complete space may not be reflexive.

UNIT—IV

13. If B and B′ are Banach spaces and T is one-one continuous linear transformation of B onto B′, then T is homeomorphism.

14. State the prove uniform boundedness principle (Banach Steinhaus theorem).

15. If B and B′ are Banach spaces and T is a linear transformation of B into B′ such that graph of T is closed then show that T is continuous.

16. If T is an operator on normed linear space N, prove that its conjugate $T^* : N^* \to N^*$ defined as $T^*(f) = f \circ T$ for all $f \in N^*$ is an operator on N^* and the mapping $\phi = \mathcal{B}(N) \to \mathcal{B}(N^*)$ defined by $\phi(T) = T^*$ preserves the norm.

UNIT—V

17. Show that a closed convex subset of a Hilbert space H contains a unique vector of smallest norm.

18. Prove that every non zero Hilbert space contains a complete orthonormal set.

19. If M is a proper closed linear subspace of a Hilbert space H, then there exists a non zero vector in H which is orthogonal to M.

20. Let H be a Hilbert space and let f be an arbitrary functional in H*. Then there exists a unique vector y in H such that

$$f = f_y \text{ i.e. } f(x) = \langle x, y \rangle \text{ for every } x \in X.$$